

Pair correlations in sandpile model: a check of logarithmic conformal field theory

V.S. Poghosyan¹, S.Y. Grigorev¹, V.B. Priezzhev¹ and P. Ruelle²

¹*Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Russia*

²*Institut de Physique Théorique, Université catholique de Louvain, B-1348 Louvain-La-Neuve, Belgium*

We compute the correlations of two height variables in the two-dimensional Abelian sandpile model. We extend the known result for two minimal heights to the case when one of the heights is bigger than one. We find that the most dominant correlation $\log r/r^4$ exactly fits the prediction obtained within the logarithmic conformal approach.

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Conformal field theory has proved to be extraordinarily powerful in the description of universality classes of equilibrium critical models in two dimensions [1]. Critical exponents, correlation functions, finite-size scaling, perturbations and boundary conditions, among others, have all been studied within the conformal approach, and thoroughly (and successfully) compared with numerical data.

More recently, increased interest has been turned toward logarithmic conformal theories, as a larger class of conformal theories, interesting in its own right, but also as a description of certain non-equilibrium lattice models. In particular, dense polymers [2, 3], sandpile models [4, 5] and percolation [3, 6] are lattice realizations of logarithmic conformal theories. An infinite series of such lattice models have been defined in [3].

The logarithmic theories are however much lesser understood than the more usual, non-logarithmic ones. This is due to their higher level of complexity, which somehow reflect the complexity of the associated lattice models. Indeed the models mentioned above all have intrinsic non-local features. In this respect, it may appear to be strange, if not miraculous, that a lattice model with non-local variables can be described, in the scaling limit, by a local field theory. The only trace the lattice non-localities leave in the continuum local theory seems to be the presence of logarithms in correlation functions.

It is therefore essential to check that the logarithmic conformal description is indeed appropriate for these models, as extensively as it has been done for equilibrium critical phenomena (see for instance [7]).

It is our purpose in this Letter to take further steps in this necessary procedure, in the context of the two-dimensional Abelian sandpile model. A certain number of checks have been carried out for this model (see [4]), but the one we propose here is more crucial because it deals with microscopic variables which are manifestly non-local, and for which the logarithmic conformal theory makes a very definite prediction. It therefore exposes in the clearest possible way the non-local features of the model.

Namely, we compute, in the infinite discrete plane, the 2-site correlations $P_{1i}(r) - P_1P_i$ of two height variables,

one of which being equal to 1, the other, h_i , being equal to 2, 3 or 4 (here P_i is the 1-site probability on the infinite plane). Conformal field theory predicts that the dominant term of these is given by [5]

$$P_{1i}(r) - P_1P_i = c_i \frac{\log r}{r^4} + \dots, \quad (1)$$

with known coefficients c_i . New and explicit lattice calculations, to be detailed below, fully confirm these results, and exactly reproduces the coefficients c_i .

Logarithmic conformal theory also predicts that the 2-site correlations $P_{ij}(r) - P_iP_j$ of two heights bigger or equal to 2 decay like $\log^2 r/r^4$, but the explicit lattice calculation of these remains out of range for the moment.

THE SANDPILE MODEL AND LOGARITHMIC CONFORMAL THEORY

We briefly recall the sandpile model introduced by Bak, Tang and Wiesenfeld in [8] (see [9] for further details).

Every site i of a finite rectangular grid \mathcal{L} is assigned a height variable h_i , taking the four values 1, 2, 3 and 4. A configuration \mathcal{C} is the set of values $\{h_i\}$ for all sites. A discrete stochastic dynamics is defined on the set of configurations. If \mathcal{C}_t is the configuration at time t , the height at a random site i of \mathcal{C}_t is incremented by 1, $h_i \rightarrow h_i + 1$, making a new configuration \mathcal{C}'_t . If the (new) height h_i in \mathcal{C}'_t is smaller or equal to 4, one simply sets $\mathcal{C}_{t+1} = \mathcal{C}'_t$. If not, all sites j such that their height variables h_j exceed 4 topple, a process by which h_j is decreased by 4, and the height of all the nearest neighbours of j are increased by 1. That is, when the site j topples, the heights are updated according to

$$h_i \rightarrow h_i - \Delta_{ji}, \quad (2)$$

with Δ the discrete Laplacian, $\Delta_{ii} = 4$, $\Delta_{ij} = -1$ for nearest neighbour sites, and $\Delta_{ij} = 0$ otherwise. This toppling process stops when all height variables are between 1 and 4; the configuration so obtained defines \mathcal{C}_{t+1} .

The boundary sites are dissipative, because a toppling there evacuates one or two grains of sand, which we imagine are collected in a sink site, connected to all dissipative

sites. The presence of dissipative sites is essential for the dynamics to be well-defined, since it makes sure that the toppling process stops in a finite time.

When the dynamics is run over long periods, the sandpile builds up, being subjected to avalanches spanning large portions of the system. This correlates the height variables over very large distances, and makes the system critical in the thermodynamic limit.

It turns out that, when the dynamics is run for long enough, and no matter what the initial configuration is, the sandpile enters a stationary regime, in which only special configurations occur with equal probability, the so-called recurrent configurations [10]. The recurrent set \mathcal{R} forms a small fraction of all configurations, since

$$|\mathcal{R}| = \det \Delta \simeq (3.21)^N, \quad (3)$$

where N is the number of sites. So the asymptotic state of the sandpile is controlled by a unique invariant distribution $P_{\mathcal{L}}^*$, uniform on the set \mathcal{R} of recurrent configurations, and zero on the non-recurrent (transient) ones. In the infinite volume limit, the invariant measure $P_{\mathcal{L}}^*$ is believed to become a conformal field theoretic measure.

To be recurrent, the height values of a configuration must satisfy certain global conditions [10], leading to non-local features. For what follows, it will be enough to know that the recurrent configurations are in one-to-one correspondence with oriented spanning trees on \mathcal{L}^* , the original lattice \mathcal{L} supplemented with the sink site. This change of variables, more convenient to perform actual calculations, also yields a different lighting on the non-localities of the model.

Spanning trees are acyclic configurations of arrows: at each site i of \mathcal{L} , there is an outgoing arrow, pointing to any one of its Δ_{ii} neighbours (if i is dissipative, the arrow can point to the sink site). A configuration of arrows defines a spanning tree if it contains no loop. By construction, the paths formed by the arrows all lead to the sink site \star , which is the root of the tree.

The mapping between recurrent configurations and trees is complicated and non-local; however the spanning trees provide an equivalent description. The global conditions that the heights of recurrent configurations have to satisfy are encoded in the property of arrow configurations of containing no loop, also a global constraint. The invariant measure $P_{\mathcal{L}}^*$ becomes simply a uniform distribution on the spanning trees.

Height values at a given site can be related to properties of spanning trees. To do so, one defines the notion of predecessor: a site j is a predecessor of i if the unique path from j to the root passes through i . Then it has been shown [11] that the trees in which the site i (not on the boundary) has exactly $a - 1$ predecessors among its nearest neighbours correspond to configurations where $h_i \geq a$, for $a = 1, 2, 3$ or 4 . So configurations with $h_i = 1$ are associated with trees which have a leaf at i ; this is a local property which may be verified by looking at the

neighbourhood of i only. In contrast, heights 2, 3 and 4 correspond to non-local properties in terms of the trees.

Using this correspondence, joint probabilities for heights $P[h_i = a, h_j = b, \dots]$ can be related to the fractions of trees satisfying certain conditions regarding the number of predecessors of i, j, \dots among their nearest neighbours. However, because of the remark we have just made, probabilities with heights 1 only are considerably easier than those involving higher heights. So far, the only probabilities involving higher heights in the bulk which have been computed are the 1-site probabilities $P[h_i = a]$ on the upper-half plane [5]. They provided enough input to assess the conformal nature of the four height variables in the scaling limit.

The logarithmic conformal theory, relevant to the sandpile model, has central charge $c = -2$. Among the distinctive features of a logarithmic theory is the presence of reducible yet indecomposable Virasoro representations; this property in turn introduces logarithms in their correlators [12].

The fields describing the scaling limit of the four lattice height variables $\delta(h_z - i) - P_i$, which we call $h_i(z)$, have been determined in [5]. As hinted by the remarks made above, the height 1 field is very different from the other heights' fields. It turns out that h_1 is a primary field with conformal weights $(1, 1)$, while the other three, h_2, h_3 and h_4 , are all related to a single field, identified with the logarithmic partner of h_1 . More precisely, if h_1 is the primary field normalized as the height 1 variable on the lattice, then h_2 satisfies the triangular relations,

$$L_0 h_2 = h_2 - \frac{1}{2} h_1, \quad L_1 h_2 = \rho, \quad L_{-1} \rho = -\frac{1}{4} h_1, \quad (4)$$

where ρ is a $(0, 1)$ field. In fact, h_1 and h_2 are members of the non-chiral version of the indecomposable representation called $\mathcal{R}_{2,1}$ in [13]. The last two fields are linear combinations, $h_3 = \alpha_3 h_2 + \beta_3 h_1$, $h_4 = \alpha_4 h_2 + \beta_4 h_1$, and may also be viewed as logarithmic partners of h_1 . The coefficients α_i, β_i are such that h_3 and h_4 , like h_1 and h_2 , have the same normalization as their lattice counterparts; their exact values are known [5].

The identification of the height fields makes it possible to compute correlations. In particular the joint probabilities for two height variables on the infinite plane \mathbb{Z}^2 correspond, in the scaling limit, to 3-point correlators in the conformal theory,

$$P_{ij}(z_1, z_2) - P_i P_j = \langle h_i(z_1) h_j(z_2) \omega(\infty) \rangle, \quad (5)$$

where ω is a weight $(0, 0)$ conformal field, logarithmic partner of the identity [5]. Indeed the infinite plane should be thought of as the limit of a growing finite grid, which has dissipation located along the boundary. In the infinite volume limit, the boundaries, and with them, the dissipation, are sent off to infinity. The field ω precisely realizes the insertion of dissipation at infinity, required for the sandpile model to be well-defined.

The 3-point correlators have been computed in [5], and take the general form ($z_{12} \equiv z_1 - z_2$)

$$\langle h_i(z_1) h_j(z_2) \omega(\infty) \rangle = \frac{A_{ij} + B_{ij} \log |z_{12}| + C_{ij} \log^2 |z_{12}|}{|z_{12}|^4}, \quad (6)$$

where $C_{ij} = 0$ if $\min(i, j) = 1$, and moreover $B_{11} = 0$ and $A_{11} = -P_1^2/2$ [14], so that, depending on i and j , one, two or three terms in the numerator are present. The coefficient of the dominant term, i.e. the largest power of $\log |z_{12}|$, could be determined exactly, and yields the dominant contribution of the 2-site probabilities [5]

$$P_{1i}(r) - P_1 P_i \simeq -\frac{\alpha_i P_1^2}{2r^4} \log r, \quad i > 1, \quad (7)$$

$$P_{ij}(r) - P_i P_j \simeq -\frac{\alpha_i \alpha_j P_1^2}{2r^4} \log^2 r, \quad i, j > 1, \quad (8)$$

where $P_1 = 2(\pi - 2)/\pi^3$, as first computed in [14], and

$$\alpha_2 = 1, \quad \alpha_3 = \frac{8 - \pi}{2(\pi - 2)}, \quad \alpha_4 = -\frac{\pi + 4}{2(\pi - 2)}. \quad (9)$$

CALCULATIONS ON THE LATTICE

It has been shown in [14] and [11] (see also [5] for details) that height probabilities P_i in the ASM can be reduced to the computation of determinants of discrete Laplacian matrices perturbed by a number of defects. The resulting matrices $\Delta' = \Delta + B$ differ from the regular Laplacian by a defect matrix B , with $B = 0$ except for a finite number of elements. Given a lattice point t_0 , non-zero elements of B related to t_0 can be marked by arrows at adjacent bonds (Fig.1). For instance, the non-

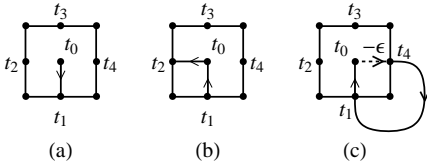


FIG. 1: Non-zero elements of B related to t_0 (arrowed bonds) for (a) $\Delta' = \Delta + B_1$, (b) $\Delta' = \Delta_{\text{local}}$ and (c) $\Delta' = \Delta_{\text{loop}}$. The bond $[t_0, t_4]$ in Δ_{loop} is weighted by $-\epsilon$.

zero part of the matrix $B = B_1$ used for the evaluation of P_1 (Fig.1a) is

$$B_1 = \begin{pmatrix} -3 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad (10)$$

where rows and columns are labeled by t_0, t_2, t_3, t_4 . The probability to have a height 1 at t_0 is then [14]

$$P_1 = \frac{\det(\Delta + B_1)}{\det \Delta} = \det(\mathbb{I} + B_1 G), \quad (11)$$

where $G = \Delta^{-1}$. The explicit form of the translation invariant Green function on the plane, $G(\vec{r}) \equiv G_{\vec{r}, \vec{0}} = G_{0,0} + g_{p,q}$ for $\vec{r} = (p, q)$,

$$g_{p,q} = \frac{1}{8\pi^2} \iint_{-\pi}^{\pi} \frac{e^{ip\alpha} e^{iq\beta} - 1}{2 - \cos \alpha - \cos \beta} d\alpha d\beta, \quad (12)$$

implies $P_1 = 2(\pi - 2)/\pi^3$.

The probability to have a height 2 can be written [11] as

$$P_2 = P_1 + \frac{4 \det \Delta_{\text{local}}}{\det \Delta} + \lim_{\epsilon \rightarrow \infty} \frac{4 \det \Delta_{\text{loop}}}{\epsilon \det \Delta} + \sum_{[a,b,c]} \lim_{\epsilon \rightarrow \infty} \frac{2 \det \Delta_{\Theta}}{\epsilon^3 \det \Delta}, \quad (13)$$

where the defect matrices related to Δ_{local} and Δ_{loop} are shown in Fig.1b-c, and that related to Δ_{Θ} is on the left side of Fig.2. The matrix Δ_{Θ} differs from Δ by the removed bond $[j_0, j_3]$ and three additional matrix elements (bonds) weighted by $-\epsilon$ between the sites j_0, j_2, j_4 and a triplet of neighbouring sites $[a, b, c]$, whose position and orientation (see Fig.3) are to be summed over the whole lattice, with the restriction that the group $[a, b, c]$ does not overlap with j_0, j_2, j_3, j_4 .

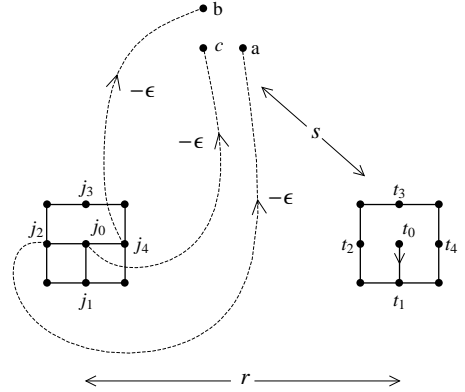


FIG. 2: Structure of matrix $\Delta_{1\Theta}$. On the left side, the defects are the removed bond $[j_3, j_0]$ and three additional bonds $[j_4, b]$, $[j_0, c]$, $[j_2, a]$ with weight $-\epsilon$. The right part shows the same defect as in Fig.1a.

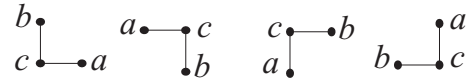


FIG. 3: Four possible orientations of the group $[a, b, c]$.

The 2-site probability P_{12} combines the defect of Δ_1 with those of Δ_{local} , Δ_{loop} and Δ_{Θ} . Simple calculations show that the matrices Δ_{local} and Δ_{loop} contribute a term $1/r^4$ to the asymptotics of $P_{12}(r)$ for large r , and are therefore subdominant. Thus the leading contribution comes from the matrix $\Delta_{1\Theta}$ combining the defects of Δ_1

and Δ_Θ , as shown in Fig.2. The correlation function $P_{1\Theta}(r)$ is

$$P_{1\Theta}(r) = \sum'_{[a,b,c]} \lim_{\varepsilon \rightarrow \infty} \frac{2 \det \Delta_{1\Theta}}{\varepsilon^3 \det \Delta}, \quad (14)$$

where the prime means that the sum excludes the terms where at least one edge in the group $[a, b, c]$ overlaps a deleted edge adjacent to t_0 . The ten forbidden positions are shown in Fig.4.

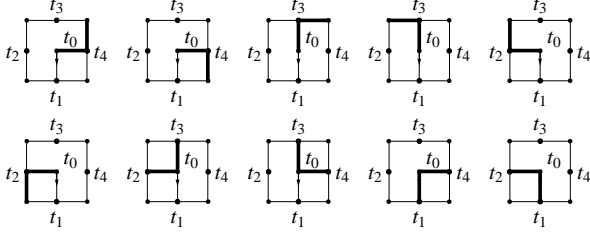


FIG. 4: Forbidden positions of the group $[a, b, c]$.

The ratios of determinants in Eq.(14) are computed as $\det(\mathbb{I} + B_{1\Theta}G)$, like Eq.(11), where the non-zero part of $B_{1\Theta} = \Delta_{1\Theta} - \Delta$ is a 8×9 block diagonal matrix. The first block is

$$B_\Theta = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\varepsilon & 0 & 0 \\ 0 & 0 & 0 & -\varepsilon & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon \end{pmatrix}, \quad (15)$$

with rows j_3, j_0, j_2, j_4 and columns j_0, j_3, c, a, b , while the second block is B_1 in Eq.(10). For large $r \gg 1$, we can replace all Green functions containing r by their asymptotic value,

$$g_{p,q} = -\frac{\ln(p^2 + q^2)}{4\pi} - \frac{1}{\pi} \left(\frac{\gamma}{2} + \frac{3}{4} \log 2 \right), \quad (16)$$

where $p^2 + q^2 \gg 1$ and $\gamma = 0.57721 \dots$ is the Euler constant. Expanding the determinants in Eq.(14), we obtain, for the sum over the forbidden positions,

$$F(r) = \frac{2(\pi - 2)^2 \log r}{\pi^6 r^4} + O\left(\frac{1}{r^4}\right). \quad (17)$$

We can now write the desired correlation in the form

$$P_{1\Theta}(r) - P_1 P_\Theta = 2 \left(\sum_{\vec{s}} \mathcal{U}_r(\vec{s}) - F(r) \right), \quad (18)$$

where the sum is taken over all lattice points $\vec{s} = (k, l)$ and P_Θ is the last term in Eq.(13). The function $\mathcal{U}_r(\vec{s})$ behaves as r^{-4} , if $s > r \gg 1$ and as $s^{-4} r^{-4} \log r$, if $r \gg 1$, $s < r$. In the region $r \gg 1$ and $s < r$, we have

$$\mathcal{U}_r(\vec{s}) = Q_{k,l} \frac{\log r}{r^4} + O\left(\frac{1}{r^4}\right), \quad (19)$$

where we find, after some algebra,

$$Q_{k,l} = \frac{(\pi - 2)^2}{4\pi^6} \left(g_{k-1,l-1} - 4g_{k-1,l} + g_{k-1,l+1} - g_{k,l-2} + 4g_{k,l} - g_{k,l+2} - g_{k+1,l-1} + 4g_{k+1,l} - g_{k+1,l+1} - 2g_{k+2,l} \right). \quad (20)$$

The summation over all k, l yields

$$\sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} Q_{k,l} = \frac{(\pi - 2)^2}{\pi^6}. \quad (21)$$

Finally, we obtain

$$P_{1\Theta}(r) - P_1 P_\Theta = -\frac{2(\pi - 2)^2 \log r}{\pi^6 r^4} + O\left(\frac{1}{r^4}\right), \quad (22)$$

which coincides with the LCFPT prediction (7) for $i = 2$. Similar calculations for $P_{13}(r)$ and $P_{14}(r)$ fully confirms the results (7) with the correct values of the coefficients.

Despite a very specific form of Δ_Θ , the correlation functions $P_{1i}(r)$, $i = 2, 3, 4$, are the first example where the logarithmic corrections to pair correlations can be computed explicitly.

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